

Calculus II Practice Final Exam

This practice test features cumulative problems based on the content in the following chapters.

[Chapter 5: Applications of Integration](#)

[Chapter 6: Integration Techniques](#)

[Chapter 7: Further Applications of Integration](#)

[Chapter 8: Differential Equations](#)

[Chapter 9: Parametric Equations and Polar Coordinates](#)

[Chapter 10: Infinite Sequences and Series](#)

(Click on a chapter to open the online textbook.)

Directions: For each question, show your work. Follow these guidelines:

- Do not use a calculator of any kind.
- Adhere to the time limit of 180 minutes.
- After you complete all the questions, score yourself according to the Solutions at the end.
Note any topics that require revision.

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Calculus II**Number of Questions—25****Suggested Time—3 hours****NO CALCULATOR****Scoring Chart**

Section	Points	Points Available
True/False Questions		22
Short Questions		33
Question 23		15
Question 24		15
Question 25		15
TOTAL		100

True/False Questions

Circle "TRUE" or "FALSE" for each statement.

1. The integral $\int_{-\pi}^{5\pi/6} \tan x \, dx$ is improper. (2 pts.)

TRUE

FALSE

2. If a nonzero function changes at a rate proportional to itself, then the function must be exponential. (2 pts.)

TRUE

FALSE

3. The parametric equations $x = -6 + 2 \sin t$ and $y = 10 - 2 \cos t$ trace out a non-circular ellipse. (2 pts.)

TRUE

FALSE

4. If u and v are positive functions with $v(x) \geq u(x)$ on $[1, \infty)$ and $\int_1^\infty u(x) \, dx$ converges, then $\int_1^\infty v(x) \, dx$ must also converge. (2 pts.)

TRUE

FALSE

5. An appropriate integrating factor for solving the differential equation (2 pts.)

$$(\sec 4t)y' + 4y = e^{-t}$$

is $\mu(t) = e^{\sin 4t}$.

TRUE

FALSE

6. The Ratio Test cannot determine whether a p -series converges or diverges. (2 pts.)

TRUE

FALSE

7. The function defined by (2 pts.)

$$f(x) = \begin{cases} \frac{3}{8}(x - \sqrt{x}) & 0 \leq x \leq 4 \\ 0 & \text{otherwise} \end{cases}$$

is a probability density function.

TRUE

FALSE

8. If $\{a_n\}_2^\infty$ converges, then so does $\left\{\frac{a_n}{\ln n}\right\}_2^\infty$. (2 pts.)

TRUE

FALSE

9. For the differential equation $\frac{dy}{dt} = 0.05y(100 - 2y)$, the fastest rate of growth occurs when $y = 50$. (2 pts.)

TRUE

FALSE

10. If a particle's acceleration is proportional to the negative of its displacement, then the displacement is a sinusoidal function of time. (2 pts.)

TRUE

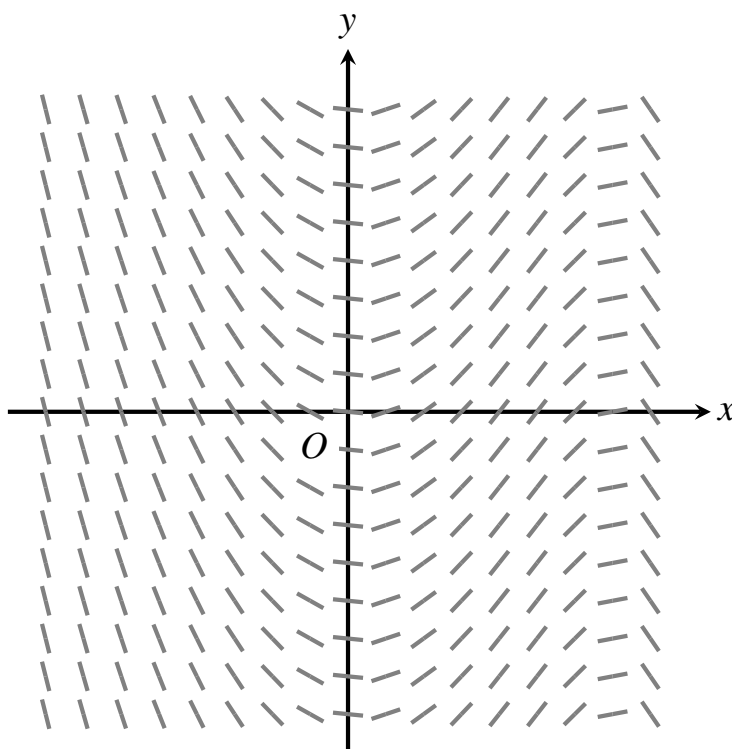
FALSE

11. The differential equation represented by the slope field below could be of the form

(2 pts.)

$$\frac{dy}{dx} = g(y)h(x),$$

where g and h are nonconstant functions.



TRUE

FALSE

Short Questions

12. Evaluate $\int (t^2 - t) e^{-t/10} dt$.

(3 pts.)

13. Determine the general solution to the differential equation $y' - x = 4xy^2$.

(3 pts.)

14. A particle follows a trajectory modeled by the parametric equations $x = e^t \sin t$ and $y = e^t \cos t$. (3 pts.)
Calculate the distance the particle travels over $0 \leq t \leq 1$.

15. A quadratic function $D(x) = 15 - Bx^2$ models the demand function for a product. If each (3 pts.)
unit is sold for \$10 and the consumer surplus is \$20, then calculate B and the number of units
demanded, X .

16. Evaluate $\int_{-4}^{-1} \frac{1}{27}(x+4)^3 \sqrt{x^2+8x+25} \, dx$.

(3 pts.)

17. A curve is traced by the parametric equations $x = 4t^2 - 8t - 16\ln|9t|$ and $y = 2t - \ln(t^4)$. Find the values of t at which the curve has horizontal tangents and vertical tangents. (3 pts.)

18. Determine a Maclaurin series for $\frac{-7x^6}{(x^7+1)^2}$. [Hint: Find a series for $\frac{1}{x^7+1}$ and differentiate the result.] (3 pts.)

19. Evaluate $\int \frac{3 \sin x \cos x}{\sin^2 x - 5 \sin x - 6} dx$. (3 pts.)

20. Prove that $\sum_{n=1}^{\infty} \frac{n + \sin 2n \cos^3 6n}{n^3}$ is convergent.

(3 pts.)

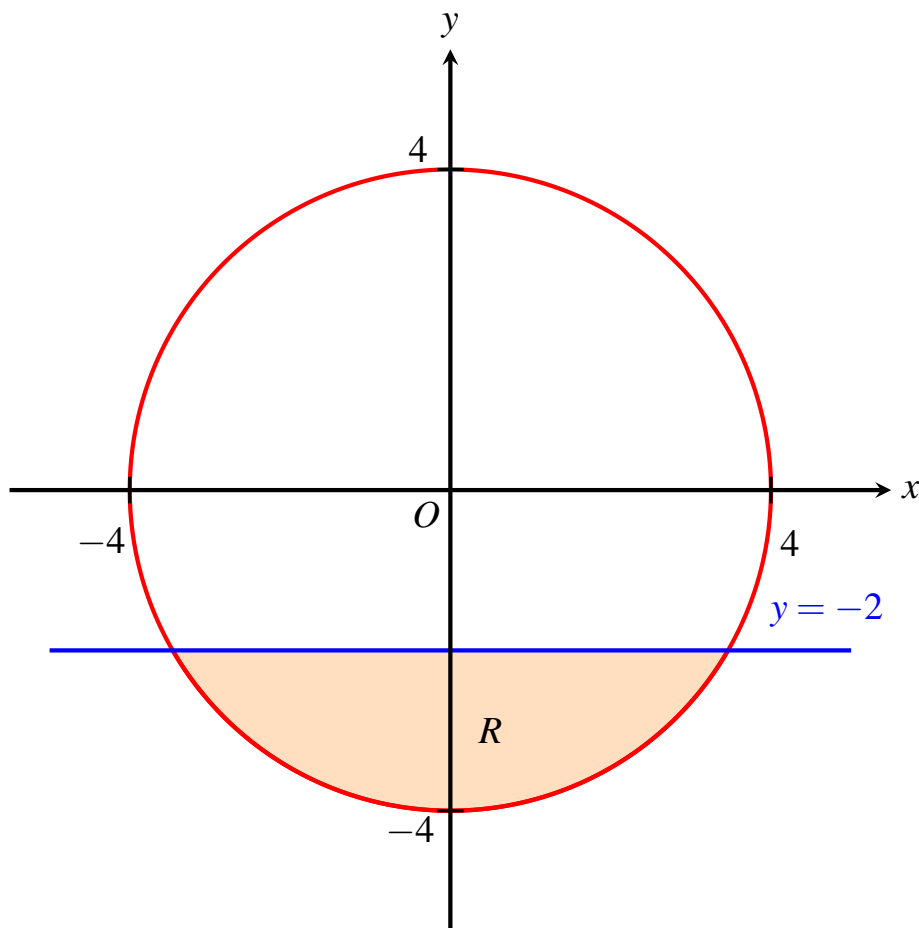
21. A liquid with a uniform density ρ completely fills a sphere of radius r . The acceleration due to gravity is g . Determine an expression for the work needed to pump all the liquid to the top in terms of r , ρ , and g .

(3 pts.)

22. A function f has derivatives of all orders. It is known that $\left|f^{(N+1)}(x)\right| \leq (N-1)!$ for all x . Using (3 pts.) Taylor's Remainder Theorem, find the largest guaranteed interval of x over which the Taylor series for f centered at a converges to f .

Long Questions

23. Region R is bounded above by the line $y = -2$ and below by a circle of radius 4 centered at the origin, as shown below. Both x and y are measured in meters.



- (a) In polar coordinates, set up an integral that equals the area of R .

(3 pts.)

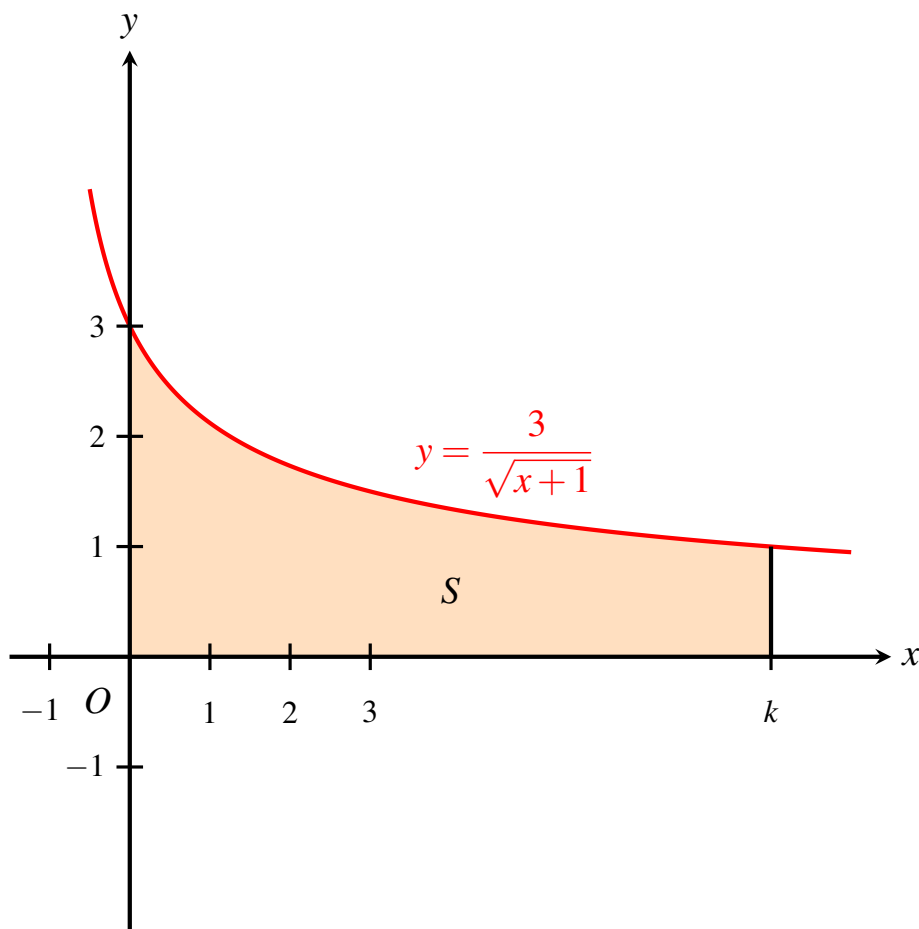
(b) With respect to the polar angle, what is the average distance from the origin to the top boundary of R ? (2 pts.)

(c) Write an integral that equals the volume of the solid generated upon rotating R about $y = 1$. (2 pts.)

(d) Set up an integral that equals the volume of the solid generated by rotating R about $x = -5$. (2 pts.)

- (e) A line $y = K$, where $-4 < K < -2$, splits R into two subregions. In Cartesian coordinates, write an expression containing integrals that equals the area of the top subregion. (2 pts.)
- (f) Suppose that water completely submerges the third and fourth quadrants of the figure, and the x -axis is the surface of the water. Write an integral that equals the hydrostatic force, in newtons, acting against R . (4 pts.)

24. Region S is bounded by the graph of $y = \frac{3}{\sqrt{x+1}}$, the x -axis, the y -axis, and the line $x = k$ (where $k > 0$), as shown below.



- (a) Write an integral that equals the arc length of $y = \frac{3}{\sqrt{x+1}}$ over $0 \leq x \leq k$.

(2 pts.)

- (b) Set up an integral with respect to x that gives the surface area of revolving $y = \frac{3}{\sqrt{x+1}}$, (2 pts.)
 $0 \leq x \leq k$, about the y -axis.

- (c) Region S is rotated about the x -axis to generate a solid. Calculate the volume of this solid as $k \rightarrow \infty$ or show that it is infinite. (3 pts.)

- (d) Find an expression for c in $[0, k]$ that satisfies the Mean Value Theorem for Integrals (2 pts.)
for $y = \frac{3}{\sqrt{x+1}}$.

(e) For $k = 15$, suppose that region S is the base of a solid whose cross sections perpendicular to the x -axis are equilateral triangles. Calculate the volume of this solid. (2 pts.)

(f) Region S now expands into the fourth quadrant and is bounded below by the line $y = L$, where $L < 0$. For $k = 8$, find L such that region S balances on the x -axis. Then calculate the x -coordinate of the centroid for this new region. (4 pts.)

25. A Taylor series $\sum_{n=1}^{\infty} \frac{(-1)^n (x-9)^n}{n}$ converges to a function $g(x)$ for all x in its interval of convergence.

(a) Find the interval of convergence and radius of convergence for $\sum_{n=1}^{\infty} \frac{(-1)^n (x-9)^n}{n}$. (4 pts.)

(b) If G satisfies $G'(x) = g(x)$ and $G(9) = 1$, then find the Taylor series for $G(x)$ about $x = 9$. (2 pts.)

(c) Calculate $g^{(6)}(9)$, the sixth derivative of g evaluated at 9. (2 pts.)

- (d) Let S_3 be the sum of the first three terms of $\sum_{n=1}^{\infty} \frac{(-1)^n (x-9)^n}{n}$ evaluated at $x = 9.1$. (3 pts.)

When S_3 is used to approximate the value of $g(9.1)$, use the Alternating Series Error Bound to find a number Q such that $S_3 - Q \leq g(9.1) \leq S_3 + Q$. Additionally, does S_3 overestimate or underestimate the true value of $g(9.1)$?

- (e) Write the Taylor series for $\sin(x-9)$ and e^{x-9} centered at $x = 9$. (2 pts.)

- (f) Use the infinite series from part (e) to evaluate $\lim_{x \rightarrow 9} \frac{1 - e^{x-9} + 2g(x)}{\sin(x-9)}$. (2 pts.)

This marks the end of the test. The solutions and scoring rubric begin on the next page.

True/False Questions (2 points each)

1. **TRUE.** The function $\tan x$ has several vertical asymptotes in $\left[-\pi, \frac{5\pi}{6}\right]$, so the integral $\int_{-\pi}^{5\pi/6} \tan x \, dx$ is improper. **

2. **TRUE.** If y changes at a rate proportional to itself, then $\frac{dy}{dx} = ky$ for some nonzero k . The solution to the differential equation is $y = y_0 e^{kx}$ where $y(0) = y_0$. **

3. **FALSE.** Rearranging $x = -6 + 2 \sin t$ and $y = 10 - 2 \cos t$ show $2 \sin t = x + 6$ and $2 \cos t = y - 10$. By the Pythagorean identity,

$$(2 \sin t)^2 + (2 \cos t)^2 = 4$$

$$(x + 6)^2 + (y - 10)^2 = 4.$$

Thus, the equations $x = -6 + 2 \sin t$ and $y = 10 - 2 \cos t$ trace out a circle of radius 2. **

4. **FALSE.** By comparison properties, if $0 \leq u(x) \leq v(x)$ on $[1, \infty)$ and $\int_1^\infty v(x) \, dx$ is convergent, then $\int_1^\infty u(x) \, dx$ is also convergent. But the test is inconclusive if $\int_1^\infty u(x) \, dx$ converges. **

5. **TRUE.** Dividing both sides of $(\sec 4t)y' + 4y = e^{-t}$ by $\sec 4t$ gives

$$y' + 4(\cos 4t)y = e^{-t} \cos 4t.$$

The integrating factor is therefore

$$\mu(t) = e^{\int 4 \cos 4t \, dt} = e^{\sin 4t}. \quad **$$

6. **TRUE.** A p -series is of the form $\sum_{n=1}^{\infty} \frac{1}{n^p}$. Letting $a_n = \frac{1}{n^p}$, we have

$$L = \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{1}{(n+1)^p} \cdot \frac{n^p}{1} \right| = \lim_{n \rightarrow \infty} \left| \frac{n^p}{(n+1)^p} \right| = 1.$$

Because $L = 1$, the Ratio Test is inconclusive.

**

7. **FALSE.** Although $\int_{-\infty}^{\infty} f(x) dx = 1$, $f(x)$ is negative on $(0, 1)$. Probability density functions must have nonnegative values, so f cannot be one.

**

8. **TRUE.** If the sequence $\{a_n\}$ converges, then $\lim_{n \rightarrow \infty} a_n = L$. Then the limit of $\left\{ \frac{a_n}{\ln n} \right\}$ is

$$\lim_{n \rightarrow \infty} \frac{a_n}{\ln n} = \frac{L}{\infty} = 0.$$

Thus, $\left\{ \frac{a_n}{\ln n} \right\}$ converges.

**

9. **FALSE.** Rewriting the differential equation gives

$$\frac{dy}{dt} = 0.1y(50 - y),$$

so the carrying capacity is $L = 50$. The maximum rate of change occurs when y reaches half the carrying capacity—that is, when $y = 25$.

**

10. **TRUE.** Let $x(t)$ be the position function (where t is time) of a particle. If x is sinusoidal, then it is of the form $x(t) = C \cos bt$ or $x(t) = C \sin bt$. For the former case $x(t) = C \cos bt$, we have

$$x''(t) = -Cb^2 \cos bt = -b^2 x(t).$$

Likewise, for the latter case $x(t) = C \sin bt$, we see

$$x''(t) = -Cb^2 \sin bt = -b^2 x(t).$$

Thus, both $x(t) = C \cos bt$ and $x(t) = C \sin bt$ are solutions to the differential equation $x''(t) = -b^2 x(t)$, which states that the acceleration $x''(t)$ is proportional to the negative of the displacement $x(t)$. **

11. **FALSE.** The slopes do not change as y is changed, so the derivative $\frac{dy}{dx}$ depends only on x . **

Short Questions (3 points each)

12. Using Integration by Parts with $u = (t^2 - t)$ and $dv = e^{-t/10} dt$, we have $du = (2t - 1) dt$ and $v = -10e^{-t/10}$. Thus,

$$\int (t^2 - t) e^{-t/10} dt = -10(t^2 - t) e^{-t/10} + 10 \int e^{-t/10} (2t - 1) dt.$$

Using Integration by Parts again with $u = (2t - 1)$ and $dv = e^{-t/10} dt$, as well as $du = 2 dt$ and $v = -10e^{-t/10}$, shows

$$\begin{aligned} \int (t^2 - t) e^{-t/10} dt &= -10(t^2 - t) e^{-t/10} - 100(2t - 1) e^{-t/10} + 200 \int e^{-t/10} dt \\ &= -10(t^2 - t) e^{-t/10} - 100(2t - 1) e^{-t/10} - 2000e^{-t/10} + C \\ &= \boxed{(-10t^2 - 190t - 1900) e^{-t/10} + C} \end{aligned}$$

13. We first rearrange $y' - x = 4xy^2$ as

$$\frac{dy}{dx} = x(4y^2 + 1).$$

Using Separation of Variables gives

$$\int \frac{1}{4y^2 + 1} dy = \int x dx.$$

To evaluate the left integral, we substitute $u = 2y$; then $du = 2 dy$ and

$$\int \frac{1}{4y^2 + 1} dy = \frac{1}{2} \int \frac{1}{u^2 + 1} du = \frac{1}{2} \tan^{-1} u = \frac{1}{2} \tan^{-1} 2y$$

(where we have omitted the constant of integration). Hence, we have

$$\frac{1}{2} \tan^{-1} 2y = \frac{1}{2} x^2 + C \quad \text{or} \quad \boxed{y = \frac{1}{2} \tan(x^2 + C)}$$

14. Differentiating $x = e^t \sin t$ and $y = e^t \cos t$ shows

$$\frac{dx}{dt} = e^t \sin t + e^t \cos t,$$

$$\frac{dy}{dt} = e^t \cos t - e^t \sin t.$$

We see

$$\begin{aligned}
 \left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 &= (e^t \sin t + e^t \cos t)^2 + (e^t \cos t - e^t \sin t)^2 \\
 &= e^{2t} \sin^2 t + 2e^t \sin t \cos t + e^{2t} \cos^2 t + e^{2t} \cos^2 t - 2e^t \sin t \cos t + e^{2t} \sin^2 t \\
 &= 2e^{2t} (\sin^2 t + \cos^2 t) = 2e^{2t}.
 \end{aligned}$$

The arc length over $0 \leq t \leq 1$ is therefore

$$\begin{aligned}
 L &= \int_0^1 \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt \\
 &= \int_0^1 \sqrt{2e^{2t}} dt \\
 &= \sqrt{2} \int_0^1 e^t dt \\
 &= \boxed{(e-1)\sqrt{2}}
 \end{aligned}$$

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15. The demand curve $D(x) = 15 - Bx^2$ intersects the line $p = 10$ when $D(X) = 10$ —namely, when

$$15 - BX^2 = 10 \implies B = \frac{5}{X^2}.$$

*

In addition, the consumer surplus is \$20, so we have

$$\int_0^X [(15 - Bx^2) - 10] dx = 20$$

$$5X - \frac{1}{3}BX^3 = 20.$$

*

Substituting $B = \frac{5}{X^2}$ shows

$$5X - \frac{1}{3} \left(\frac{5}{X^2} \right) X^3 = 20$$

$$\frac{10}{3}X = 20$$

$$\implies X = 6.$$

Then $B = \frac{5}{6^2} = \frac{5}{36}$. Hence, the values are

$$\boxed{B = \frac{5}{36}} \quad \boxed{X = 6}$$

*

16. The expression $\sqrt{x^2 + 8x + 25}$ is equivalent to $\sqrt{(x+4)^2 + 9}$, so

$$\int_{-4}^{-1} \frac{1}{27} (x+4)^3 \sqrt{x^2 + 8x + 25} dx = \frac{1}{27} \int_{-4}^{-1} (x+4)^3 \sqrt{(x+4)^2 + 9} dx.$$

We substitute

$$x + 4 = 3 \tan \theta \implies dx = 3 \sec^2 \theta d\theta,$$

where $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$. When $x = -4$, we have $0 = 3 \tan \theta = 0 \implies \theta = 0$. Conversely, when $x = -1$, $3 = 3 \tan \theta \implies \theta = \frac{\pi}{4}$. Thus, the integral becomes

$$\begin{aligned} \frac{1}{27} \int_0^{\pi/4} (3 \tan \theta)^3 \sqrt{9 \tan^2 \theta + 9} (3 \sec^2 \theta) d\theta &= 9 \int_0^{\pi/4} \tan^3 \theta \sec^3 \theta d\theta \\ &= 9 \int_0^{\pi/4} \sec^2 \theta \tan^2 \theta (\sec \theta \tan \theta) d\theta \\ &= 9 \int_0^{\pi/4} \sec^2 \theta (\sec^2 \theta - 1) (\sec \theta \tan \theta) d\theta \end{aligned}$$

*

To strip the extra factor $\sec \theta \tan \theta$, we substitute $u = \sec \theta$; then $du = \sec \theta \tan \theta d\theta$. When $\theta = 0$, $u = 1$; when $\theta = \frac{\pi}{4}$, $u = \sqrt{2}$. The integral therefore becomes

$$\begin{aligned} 9 \int_1^{\sqrt{2}} u^2 (u^2 - 1) du &= 9 \int_1^{\sqrt{2}} (u^4 - u^2) du \\ &= \left(\frac{9}{5} u^5 - 3u^3 \right) \Big|_1^{\sqrt{2}} \\ &= \boxed{\frac{6 + 6\sqrt{2}}{5}} \end{aligned}$$

*

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17. Differentiating $x = 4t^2 - 8t - 16\ln|9t|$ and $y = 2t - \ln(t^4)$ gives

$$\frac{dx}{dt} = 8t - 8 - \frac{16}{t},$$

$$\frac{dy}{dt} = 2 - \frac{4}{t}.$$

Then

$$\begin{aligned}\frac{dy}{dx} &= \frac{dy/dt}{dx/dt} \\ &= \frac{2 - 4/t}{8t - 8 - 16/t} \\ &= \frac{2t - 4}{8t^2 - 8t - 16} \\ &= \frac{2(t - 2)}{8(t - 2)(t + 1)} \\ &= \frac{1}{4(t + 1)}.\end{aligned}$$

There is a vertical tangent when $t = -1$ because the denominator is 0 and the numerator is nonzero. However, no horizontal tangents exist because the numerator is never 0. The answers are therefore as follows:

horizontal tangents: none

vertical tangents: $t =$ -1

18. Since $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$ for $-1 < x < 1$, we have

$$\frac{1}{x^7 + 1} = \frac{1}{1 - (-x^7)} = \sum_{n=0}^{\infty} (-x^7)^n = \sum_{n=0}^{\infty} (-1)^n x^{7n}.$$

Hence,

$$\begin{aligned}\frac{-7x^6}{(x^7+1)^2} &= \frac{d}{dx} \sum_{n=0}^{\infty} (-1)^n x^{7n} \\ &= \boxed{\sum_{n=1}^{\infty} 7n(-1)^n x^{7n-1}}, \quad -1 < x < 1.\end{aligned}$$

where the starting index was incremented to $n = 1$ to avoid any 0 terms.

19. Substituting $u = \sin x$ gives $du = \cos x dx$, so the integral becomes

$$\int \frac{3u}{u^2 - 5u - 6} du = \int \frac{3u}{(u-6)(u+1)} du.$$

By partial fraction decomposition,

$$\begin{aligned}\int \frac{3u}{(u-6)(u+1)} du &= \int \left(\frac{18/7}{u-6} + \frac{3/7}{u+1} \right) du \\ &= \frac{18}{7} \ln|u-6| + \frac{3}{7} \ln|u+1| + C \\ &= \boxed{\frac{18}{7} \ln|\sin x - 6| + \frac{3}{7} \ln|\sin x + 1| + C}\end{aligned}$$

20. Note that

$$\frac{n + \sin 2n \cos^3 6n}{n^3} = \frac{1}{n^2} + \frac{\sin 2n \cos^3 6n}{n^3}.$$

Of course, $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges because it is a p -series with $p = 2 > 1$. On the other hand, note that $-1 \leq \sin 2n \cos^3 6n \leq 1$, so

$$0 \leq \frac{|\sin 2n \cos^3 6n|}{n^3} \leq \frac{1}{n^3}.$$

The series $\sum_{n=1}^{\infty} \frac{1}{n^3}$ converges because it is a p -series with $p = 3 > 1$, so by the Direct Comparison Test,

$\sum_{n=1}^{\infty} \frac{|\sin 2n \cos^3 6n|}{n^3}$ also converges. Hence, $\sum_{n=1}^{\infty} \frac{\sin 2n \cos^3 6n}{n^3}$ converges because it is absolutely

convergent. This analysis shows that $\sum_{n=1}^{\infty} \frac{n + \sin 2n \cos^3 6n}{n^3}$ converges because it is the sum of the

convergent series $\sum_{n=1}^{\infty} \frac{1}{n^2}$ and $\sum_{n=1}^{\infty} \frac{\sin 2n \cos^3 6n}{n^3}$.

21. Consider an axis system whose origin is the center of the sphere. Any thin horizontal slice of the sphere

at y has a circular cross section of radius $\sqrt{r^2 - y^2}$, so its cross-sectional area is $A(y) = \pi(r^2 - y^2)$. In addition, the slice at y is located a distance $r - y$ from the top of the sphere. Water fills the sphere from $y = -r$ to $y = r$, so the work done to pump out all the liquid is

$$\begin{aligned} W &= \rho g \int_{-r}^r A(y) dy \\ &= \rho g \int_{-r}^r \pi(r^2 - y^2)(r - y) dy \\ &= \rho g \pi \int_{-r}^r (r^3 - r^2 y - ry^2 + y^3) dy \\ &= \rho g \pi \int_{-r}^r (r^3 - ry^2) dy, \end{aligned}$$

where the last step is true because $\int_{-r}^r r^2 y dy = 0$ and $\int_{-r}^r y^3 dy = 0$ because the integrands are odd. Antidifferentiating, noting that r is a constant, we have

$$\begin{aligned} W &= \rho g \pi \left(r^3 y - \frac{1}{3} r y^3 \right) \Big|_{-r}^r \\ &= \boxed{\frac{4}{3} \rho g \pi r^4} \end{aligned}$$

- 22.** Consider an N th degree Taylor polynomial centered at a . A Taylor series converges to f for x in some interval if and only if the remainder R_N converges to 0 as $N \rightarrow \infty$. Taylor's Remainder Theorem gives the error bound

$$|R_N| \leq \frac{M}{(N+1)!} |x - a|^{N+1},$$

where $|f^{(N+1)}(x)| \leq M$ between a and x . Given $|f^{(N+1)}(x)| \leq (N+1)!$, we have

$$|R_N| \leq \frac{(N+1)!}{(N+1)!} |x - a|^{N+1} = |x - a|^{N+1}.$$

For $x = a$, $R_N = 0$ is obvious. But for all $x \neq a$, $R_N \rightarrow 0$ as $N \rightarrow \infty$ for $|x - a| \leq 1 \iff a - 1 \leq x \leq a + 1$. Hence, the largest guaranteed interval over which the Taylor series converges to f is

$$\boxed{[a - 1, a + 1]}$$

Long Questions (15 points each)

23. (a) The circle is expressed in polar coordinates as $r = 4$, and the line $y = -2$ is expressed as

$$r \sin \theta = -2 \implies r = -2 \csc \theta.$$

The points of intersection occur when

$$-2 \csc \theta = 4 \implies \theta = \frac{7\pi}{6}, \frac{11\pi}{6}.$$

Hence, the area of region R is

$$A = \frac{1}{2} \int_{7\pi/6}^{11\pi/6} \left[(4)^2 - (-2 \csc \theta)^2 \right] d\theta = \boxed{\frac{1}{2} \int_{7\pi/6}^{11\pi/6} (16 - 4 \csc^2 \theta) d\theta}$$

Alternatively, by symmetry, other valid expressions are

$$A = \int_{7\pi/6}^{3\pi/2} (16 - 4 \csc^2 \theta) d\theta = \int_{3\pi/2}^{11\pi/6} (16 - 4 \csc^2 \theta) d\theta.$$

- (b) The top boundary of region R is the line $y = -2$, or $r = -2 \csc \theta$. The average distance from $r = -2 \csc \theta$ to the origin is the average value of $r = -2 \csc \theta$ over $\left[\frac{7\pi}{6}, \frac{11\pi}{6} \right]$ —namely,

$$r_{\text{avg}} = \frac{1}{\frac{11\pi}{6} - \frac{7\pi}{6}} \int_{7\pi/6}^{11\pi/6} (-2 \csc \theta) d\theta$$

$$= -\frac{3}{\pi} \int_{7\pi/6}^{11\pi/6} \csc \theta d\theta$$

$$= \frac{3}{\pi} \ln |\csc \theta + \cot \theta| \Big|_{7\pi/6}^{11\pi/6}$$

$$= \boxed{\frac{3}{\pi} \ln \left| \frac{-2 - \sqrt{3}}{-2 + \sqrt{3}} \right|}$$

- (c) The outer boundary of region R is the curve $y = -\sqrt{16 - x^2}$, which intersects the line $y = -2$ when

$$-2 = -\sqrt{16 - x^2} \implies x = \pm \sqrt{12}.$$

The distance between $y = 1$ and the outer boundary of R is $r_{\text{out}}(x) = 1 + \sqrt{16 - x^2}$, and the distance between $y = 1$ and the inner boundary is $r_{\text{in}}(x) = 1 - (-2) = 3$. Hence, by the Washer Method the volume is

$$V = \pi \int_{-\sqrt{12}}^{\sqrt{12}} \left([r_{\text{out}}(x)]^2 - [r_{\text{in}}(x)]^2 \right) dx \quad *$$

$$= \pi \int_{-\sqrt{12}}^{\sqrt{12}} \left[\left(1 + \sqrt{16 - x^2} \right)^2 - 3^2 \right] dx \quad *$$

- (d) The horizontal distance from the line $x = -5$ to any point in R is $r(x) = x + 5$. In addition, the height of R at any x is $h(x) = -2 + \sqrt{16 - x^2}$. Hence, by the Shell Method,

$$V = 2\pi \int_{-\sqrt{12}}^{\sqrt{12}} r(x)h(x) dx \quad *$$

$$= 2\pi \int_{-\sqrt{12}}^{\sqrt{12}} (x + 5) \left(-2 + \sqrt{16 - x^2} \right) dx \quad *$$

- (e) The easiest way to express the area of the top subregion is to integrate with y . Taking a horizontal slice at y between $x = \sqrt{16 - y^2}$ and $x = -\sqrt{16 - y^2}$ gives a width of

$$w(y) = \sqrt{16 - y^2} - \left(-\sqrt{16 - y^2} \right) = 2\sqrt{16 - y^2}. \quad *$$

In addition, the top subregion is bounded between $y = K$ and $y = -2$, so its area is

$$A = \int_K^{-2} w(y) dy = \int_K^{-2} 2\sqrt{16 - y^2} dy \quad *$$

- (f) From part (e), the width of a horizontal strip of region R is

$$w(y) = 2\sqrt{16 - y^2}. \quad *$$

At any y , a horizontal strip is located a distance of

$$h(y) = -y \quad *$$

from the x -axis (the surface of the water). Hence, the hydrostatic force acting against R is, in

newtons,

$$F = 9800 \int_{-4}^{-2} h(y)w(y) \, dy$$

*

$$= \boxed{-19,600 \int_{-4}^{-2} y\sqrt{16-y^2} \, dy}$$

*

24. (a) The derivative of $y = \frac{3}{\sqrt{x+1}}$ is

$$\frac{dy}{dx} = -\frac{3}{2}(x+1)^{-3/2}.$$

So the arc length is

$$L = \int_0^k \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx$$

$$= \int_0^k \sqrt{1 + \frac{9}{4(x+1)^3}} dx$$

- (b) The surface area of revolution when $y = \frac{3}{\sqrt{x+1}}$, $0 \leq x \leq k$, is revolved around the y-axis is

$$S = 2\pi \int_0^k x ds = 2\pi \int_0^k x \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx$$

$$= 2\pi \int_0^k x \sqrt{1 + \frac{9}{4(x+1)^3}} dx$$

- (c) By the Disk Method, the volume of the solid of revolution is

$$V = \pi \int_0^k \frac{9}{x+1} dx.$$

As $k \rightarrow \infty$,

$$V = \lim_{k \rightarrow \infty} \pi \int_0^k \frac{9}{x+1} dx = \lim_{k \rightarrow \infty} 9\pi \ln|x+1| \Big|_0^k$$

$$= \lim_{k \rightarrow \infty} 9\pi \ln(k+1) = \infty.$$

Thus, the volume is infinite.

- (d) The average value of $y = \frac{3}{\sqrt{x+1}}$ on $[0, k]$ is

$$f_{\text{avg}} = \frac{1}{k-0} \int_0^k \frac{3}{\sqrt{x+1}} dx = \frac{6}{k} \sqrt{x+1} \Big|_0^k = \frac{6\sqrt{k+1} - 6}{k}.$$

The Mean Value Theorem for Integrals guarantees a number c in $[0, k]$ such that $f(c) = f_{\text{avg}}$. We have

$$\frac{3}{\sqrt{c+1}} = \frac{6\sqrt{k+1}-6}{k}$$

$$\Rightarrow c = \boxed{\frac{k^2}{(2\sqrt{k+1}-2)^2} - 1}$$

(e) At each x , a cross section has an area of

$$A(x) = \frac{\sqrt{3}}{4} \left(\frac{3}{\sqrt{x+1}} \right)^2 = \frac{9\sqrt{3}}{4(x+1)}.$$

So the volume is

$$V = \int_0^{15} \frac{9\sqrt{3}}{4(x+1)} dx$$

$$= \frac{9\sqrt{3}}{4} \ln|x+1| \Big|_0^{15}$$

$$= \boxed{\frac{9\sqrt{3}}{4} \ln 16}$$

(f) Region S is now bounded above by $y = \frac{3}{\sqrt{x+1}}$ and below by $y = L$, where $L < 0$, for $0 \leq x \leq 8$. The region balances on the x -axis if the y -coordinate of the centroid is 0. If A is the area of region S , we have

$$\bar{y} = \frac{1}{2A} \int_0^8 \left[\left(\frac{3}{\sqrt{x+1}} \right)^2 - L^2 \right] dx = 0$$

$$(9 \ln|x+1| - xL^2) \Big|_0^8 = 0$$

$$(9 \ln 9 - 8L^2) = 0$$

$$\Rightarrow L = \boxed{-\sqrt{\frac{9 \ln 9}{8}}}$$

In addition, the area is

$$A = \int_0^8 \left(\frac{3}{\sqrt{x+1}} - L \right) dx = \left(6\sqrt{x+1} + x\sqrt{\frac{9\ln 9}{8}} \right) \Big|_0^8 = 12 + 8\sqrt{\frac{9\ln 9}{8}} = 12 + \sqrt{72\ln 9}.$$

Now the x -coordinate of the centroid is

$$\begin{aligned} \bar{x} &= \frac{1}{A} \int_0^8 x \left(\frac{3}{\sqrt{x+1}} - L \right) dx \\ &= \frac{1}{12 + \sqrt{72\ln 9}} \int_0^8 \left(\frac{3x}{\sqrt{x+1}} + x\sqrt{\frac{9\ln 9}{8}} \right) dx \\ &= \frac{1}{12 + \sqrt{72\ln 9}} \left(\int_0^8 \frac{3x}{\sqrt{x+1}} dx + 32\sqrt{\frac{9\ln 9}{8}} \right) \\ &= \frac{1}{12 + \sqrt{72\ln 9}} \left(\int_0^8 \frac{3x}{\sqrt{x+1}} dx + \sqrt{1152\ln 9} \right). \end{aligned}$$

With $u = x + 1$, we see

$$\int_0^8 \frac{3x}{\sqrt{x+1}} dx = \int_1^9 \frac{3(u-1)}{\sqrt{u}} du = 3 \int_1^9 (u^{1/2} - u^{-1/2}) du = \left(2u^{3/2} - 6u^{1/2} \right) \Big|_1^9 = 40.$$

Thus,

$$\bar{x} = \frac{40 + \sqrt{1152\ln 9}}{12 + \sqrt{72\ln 9}}$$

25. (a) Letting $a_n = \frac{(-1)^n (x-9)^n}{n}$, we let

$$\begin{aligned} L &= \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| \\ &= \lim_{n \rightarrow \infty} \left| \frac{(-1)^{n+1} (x-9)^{n+1}}{n+1} \cdot \frac{n}{(-1)^n (x-9)^n} \right| \\ &= \lim_{n \rightarrow \infty} \left| \frac{(x-9)n}{n+1} \right| \\ &= |x-9| \lim_{n \rightarrow \infty} \left| \frac{n}{n+1} \right| \\ &= |x-9|. \end{aligned}$$

By the Ratio Test, the series $\sum_{n=1}^{\infty} \frac{(-1)^n (x-9)^n}{n}$ converges if $L < 1$ —that is, for

$$|x-9| < 1 \iff 8 < x < 10.$$

Substituting $x = 8$, we have

$$\sum_{n=1}^{\infty} \frac{(-1)^n (8-9)^n}{n} = \sum_{n=1}^{\infty} \frac{(-1)^{2n}}{n} = \sum_{n=1}^{\infty} \frac{1}{n},$$

which diverges because it is the Harmonic series. Conversely, at $x = 10$ we obtain

$$\sum_{n=1}^{\infty} \frac{(-1)^n (10-9)^n}{n} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n},$$

which converges by the Alternating Series Test because its terms are decreasing in magnitude to 0.

Hence, the answers are as follows:

interval of convergence: $(8, 10]$

radius of convergence: $R = 1$

(b) Since $G'(x) = g(x)$, G is an antiderivative of g . Hence, a Taylor series for G is given by

$$\begin{aligned} \int \sum_{n=1}^{\infty} \frac{(-1)^n (x-9)^n}{n} dx &= \sum_{n=1}^{\infty} \frac{(-1)^n}{n} \int (x-9)^n dx \\ &= C + \sum_{n=1}^{\infty} \frac{(-1)^n (x-9)^{n+1}}{n(n+1)} \end{aligned}$$

for some constant C . Substituting $G(9) = 1$ shows

$$1 = C + \sum_{n=1}^{\infty} \frac{(-1)^n (9-9)^{n+1}}{n(n+1)} \implies 1 = C + 0 \implies C = 1.$$

Accordingly, the Taylor series for G is

$$1 + \sum_{n=1}^{\infty} \frac{(-1)^n (x-9)^{n+1}}{n(n+1)}$$

(c) A Taylor series of g centered at 9 is, by definition, of the form $\sum_{n=0}^{\infty} \frac{g^{(n)}(9)}{n!} (x-9)^n$. Therefore, the given series $\sum_{n=1}^{\infty} \frac{(-1)^n (x-9)^n}{n}$ is equivalent to $\sum_{n=0}^{\infty} \frac{g^{(n)}(9)}{n!} (x-9)^n$ for x in the interval of convergence. Equating the sixth terms of each series, we find

$$\frac{g^{(6)}(9)}{6!} (x-9)^6 = \frac{(-1)^6 (x-9)^6}{6}$$

$$\implies g^{(6)}(9) = \frac{6!}{6} = 5! = \boxed{120}$$

(d) By the Alternating Series Error Bound, the magnitude of the error in the approximation is no greater than the magnitude of the fourth term:

$$\left| \frac{(-1)^4 (9.1-9)^4}{4} \right| = \frac{0.1^4}{4} = 2.5 \times 10^{-5}$$

We thus have

$$|g(9.1) - S_3| \leq 2.5 \times 10^{-5} \iff -2.5 \times 10^{-5} \leq g(9.1) - S_3 \leq 2.5 \times 10^{-5}.$$

Adding S_3 to each term gives

$$S_3 - 2.5 \times 10^{-5} \leq g(9.1) \leq S_3 + 2.5 \times 10^{-5}.$$

Hence,

$$Q = \boxed{2.5 \times 10^{-5}}$$

In addition, because the fourth term is positive, S_3 must underestimate the true value of $g(9.1)$.

(e) The power series for $\sin x$ and e^x are, respectively,

$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!},$$

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}.$$

Replacing x with $x - 9$ gives

$$\sin(x-9) = \sum_{n=0}^{\infty} \frac{(-1)^n (x-9)^{2n+1}}{(2n+1)!}$$

$$e^{x-9} = \sum_{n=0}^{\infty} \frac{(x-9)^n}{n!}$$

(f) The first few terms of the series for $g(x)$ are

$$-(x-9) + \frac{1}{2}(x-9)^2 - \frac{1}{3}(x-9)^3 + \dots$$

Writing out some terms, we see that $\lim_{x \rightarrow 9} \frac{1 - e^{x-9} + 2g(x)}{\sin(x-9)}$ becomes

$$\lim_{x \rightarrow 9} \frac{1 - [1 + (x-9) + \frac{1}{2}(x-9)^2 + \dots] + 2[-(x-9) + \frac{1}{2}(x-9)^2 - \frac{1}{3}(x-9)^3 + \dots]}{(x-9) - \frac{1}{3!}(x-9)^3 + \frac{1}{5!}(x-9)^5 + \dots}.$$

Simplifying, we have

$$\lim_{x \rightarrow 9} \frac{[-(x-9) - \frac{1}{2}(x-9)^2 - \dots] + [-2(x-9) + (x-9)^2 - \frac{2}{3}(x-9)^3 + \dots]}{(x-9) - \frac{1}{3!}(x-9)^3 + \frac{1}{5!}(x-9)^5 + \dots}.$$

Dividing the numerator and denominator each by $(x - 9)$ gives

$$\lim_{x \rightarrow 9} \frac{[-1 - \frac{1}{2}(x-9) - \dots] + [-2 + (x-9) - \frac{2}{3}(x-9)^2 + \dots]}{1 - \frac{1}{3!}(x-9)^2 + \frac{1}{5!}(x-9)^4 + \dots}.$$

Substituting $x = 9$ then zeroes out all the powers of $(x - 9)$, leaving

$$\frac{[-1 + (0)] + [-2 + (0)]}{1} = \boxed{-3}$$

*